It will be remembered that a century ago the problem of computing the orbit of a comet was much discussed by mathematicians and astronomers, and almost every eminent man tried his hand on this problem. Kepler discussed this question, and assumed that the comets moved in right lines; and as he never wanted for an hypothesis, he also assumed that these erratic bodies were generated in the celestial spaces, and then were destroyed in some mysterious manner. Newton's discovery of the law of gravitation showed that the rectilinear motion of a comet was impossible, but this motion continued to be assumed for small portions of the orbit, and was used by Newton himself as a first approximation. It is worthy of notice that in the memoir refered to above, Lambert divides the chord MN at the point q in such a manner that the segments of this chord are proportional to the intervals of time between the observations. This assumption was afterward adopted by Olbers in his method of computing the orbit of a comet, a method which is now almost the only one in use.

AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

BY CHAS. H. KUMMELL, ASSISTANT U. S. LAKE SURVEY, DETROIT, MICH.

(*Continued from page* 46.)

§ 6.

THERE are other applications, more or less direct, of which a few examples may be given.

Ex. 6. Assume in (6) $f(z) = e^{-c^2z^2} = e^{-c^2(x+yi)^2} = e^{-c^2(x^2-y^2)}(\cos 2c^2xy - i\sin 2c^2xy)$.

Since $e^{-c^2z^2} = \infty$ if $z = 0 \pm \infty i$ there would be a point of discontinuity if 0 was within the x-integration and $\pm\infty$ the upper or lower limit of the y-integration. But the correction would be 0 nevertheless for we have by $(10''_3)$

$$\varDelta_{0\pm\infty i} = \pi i\left[\frac{z\mp\infty i}{e^{c^2z^2}}\right]_{z=\pm\infty i} = \pi i\left[\frac{1}{2c^2z\,e^{c^2z^2}}\right]_{z=\pm\infty i} = 0.$$

There is consequently no correction, and we have by (6)

$$\int_{x_0}^{x_n} dx[e^{-c^2(x^2-y_n^2)}(\cos 2c^2xy_n - i\sin 2c^2xy_n) - e^{-c^2(x^2-y_0^2)}(\cos 2c^2xy_0 - i\sin 2c^2xy_0)]$$
$$= i\int_{y_0}^{y_n} dy[e^{-c^2(x_n^2-y^2)}(\cos 2c^2x_ny - i\sin 2c^2x_ny) - e^{-c^2(x_0^2-y^2)}(\cos 2c^2x_0y - i\sin 2c^2x_0y)]. \quad (g)$$

Let $x_0 = 0 = y_0$, then

$$\int_0^{x_n} dx\left[e^{-c^2(x^2-y_n^2)}(\cos 2c^2xy_n - i\sin 2c^2xy_n) - e^{-c^2x^2}\right]$$
$$= i\int_0^{y_n} dy\left[e^{-c^2(x_n^2-y^2)}(\cos 2c^2x_ny - i\sin 2c^2x_ny) - e^{c^2y^2}\right], \quad (g')$$

which gives the following two relations:

$$\int_0^{x_n} dx\left[e^{-c^2(x^2-y_n^2)}(\cos 2c^2xy_n - e^{-c^2x^2}\right] = \int_0^{y_n} dy\, e^{-c^2(x_n^2-y^2)}\sin 2c^2x_ny, \quad (g'_1)$$

$$\int_0^{x_n} dx\, e^{-c^2(x^2-y_n^2)}\sin 2c^2xy_n = -\int_0^{y_n} dy\, e^{c^2y^2}\left[e^{-c^2x_n^2}\cos 2c^2x_ny - 1\right]. \quad (g'_2)$$

In this let $x_n = \infty$, then

$$\int_0^{\infty} dx\left[e^{-c^2(x^2-y_n^2)}\cos 2c^2xy_n - e^{-c^2x^2}\right] = 0, \quad (g''_1)$$

$$\int_0^{\infty} dx\, e^{-c^2(x^2-y_n^2)}\sin 2c^2xy_n = \int_0^{y_n} dy\, e^{c^2y^2}. \quad (g''_2)$$

We have $\int_0^{\infty} dx\, e^{-c^2x^2} = \frac{\pi^{\frac{1}{2}}}{2c}$ a well known integral, therefore from (g''_1)

$$\int_0^{\infty} dx\, e^{-c^2x^2}\cos 2c^2xy_n = \frac{\pi^{\frac{1}{2}}}{2c}e^{-c^2y_n^2}. \quad (g''_3)$$

Ex. 7. Let $f(z) = \frac{\varphi(z)}{e^z \pm e^{-z}}$, then we have in (6)

$$\int_{x_0}^{x_n} dx\left[\frac{\varphi(x+y_ni)}{e^{x+y_ni}\pm e^{-x-y_ni}} - \frac{\varphi(x+y_0i)}{e^{x+y_0i}\pm e^{-x-y_0i}}\right] = i\int_{y_0}^{y_n} dy\left[\frac{\varphi(x_n+yi)}{e^{x_n+yi}\pm e^{-x_n-yi}}\right.$$
$$\left. - \frac{\varphi(x_0+yi)}{e^{x_0+yi}\pm e^{-x_0-yi}}\right] - \varDelta. \quad (h)$$

Let $x_n = \infty$; $x_0 = -\infty$, then $\frac{\varphi(\pm\infty+yi)}{e^{\pm\infty+yi}\pm e^{\mp\infty+yi}} = 0$ if $\varphi(\pm\infty + yi) = 0$ or a finite quantity, or an infinity of a lower order than e^{∞}, and we have from (h)

$$\int_{-\infty}^{\infty} dx\left[\frac{\varphi(x+y_ni)}{e^{x+y_ni}\pm e^{-x-y_ni}} - \frac{\varphi(x+y_0i)}{e^{x+y_0i}\pm e^{-x-y_0i}}\right] = -\varDelta. \quad (h')$$

Let us suppose $\varphi(z)$ continuous (if not, we may subsequently correct for it), then the points of discontinuity are given by the equation

$$e^z \pm e^{-z} = 0. \quad (i)$$

1. For the upper sign the roots are

$$z_{\pm\frac{1}{2}} = \pm\frac{\pi}{2}i;\; z_{\pm\frac{3}{2}} = \pm\frac{3\pi}{2}i;\; z_{\pm\frac{5}{2}} = \pm\frac{5\pi}{2}i;\ldots\; z_{\pm\frac{2n-1}{2}} = \pm\frac{(2n-1)\pi}{2}i;\ldots \quad (k)$$

The correction for discontinuity due the general value of these roots, as formula (10) gives it, is

$$\Delta_{\pm\frac{2n-1}{2}\pi i} = 2\pi i\,\varphi\left(\pm\frac{2n-1}{2}\,\pi i\right)\left[\frac{z\mp\frac{1}{2}(2n-1)\pi i}{e^z+e^{-z}}\right]_{z=\pm\frac{2n-1}{2}\pi i}$$

$$= 2\pi i\,\varphi\left(\pm\frac{2n-1}{2}\,\pi i\right).\frac{1}{e^{\pm\frac{1}{2}(2n-1)\pi i}-e^{\mp\frac{1}{2}(2n-1)\pi i}}$$

$$= \mp\,\pi e^{n\pi i}\varphi[\pm\tfrac{1}{2}(2n-1)\pi i]. \qquad (l)$$

By virtue of $(10_3{}')$ one half this correction is taken if the discontinuous element coincides with one of the limits of the y-integration.

Let $y_n = n\pi$ and $y_0 = 0$, then by (h')

$$\int_{-\infty}^{\infty} dx\left[\frac{\varphi(x+n\pi i)}{e^{x+n\pi i}+e^{-x-n\pi i}}-\frac{\varphi(x)}{e^x+e^{-x}}\right] = \pi\sum_1^n\left\{e^{s\pi i}\varphi\left(\frac{2s-1}{2}\,\pi i\right)\right\},$$

or, since $e^{n\pi i} = e^{-n\pi i} = (-1)^n$,

$$\int_{-\infty}^{\infty} dx\,\frac{e^{n\pi i}\,\varphi(x+n\pi i)-\varphi(x)}{e^x+e^{-x}} = \pi\sum_1^n\left\{e^{s\pi i}\varphi\left(\frac{2s-1}{2}\,\pi i\right)\right\}. \qquad (m)$$

Let $\varphi(z) = e^{-c^2z^2}$ then since $\dfrac{e^{-c^2(\pm\infty+yi)^2}}{e^{\pm\infty+yi}+e^{\mp\infty-yi}} = 0$ we can use (m). We have

$$\int_{-\infty}^{\infty} dx\,\frac{e^{n\pi i-c^2(x+n\pi i)^2}}{e^x+e^{-x}} = \pi\sum_1^n\left\{e^{s\pi i+c^2\left(\frac{2s-1}{2}\right)^2\pi^2}\right\}. \qquad (n)$$

Separating the possible from the impossible, we have

$$\int_{-\infty}^{\infty}\frac{e^{-c^2x^2}dx}{e^x+e^{-x}}\left[e^{c^2n^2k^2}\,{}^*\cos[n\pi(1-2c^2x)]-1\right] = \pi\sum_1^n\left\{e^{ski+c^2\left(\frac{2s-1}{2}\right)^2k^2}\right\}, \qquad (n_1')$$

$$\int_{-\infty}^{\infty}\frac{e^{-c^2x^2}dx}{e^x+e^{-x}}\,e^{c^2n^2k^2}\sin[n\pi(1-2c^2x)] = 0, \qquad (n_2')$$

or let $\varphi(z) = e^{bzi}$ then $\dfrac{e^{b(\pm\infty i-y)}}{e^{\pm\infty+yi}+e^{\mp\infty-yi}} = 0$, and from (m)

$$\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}\left[e^{nki+b(x+nki)i}-e^{bxi}\right] = \pi\sum_1^n\left\{e^{ski-\frac{2s-1}{2}bk}\right\}. \qquad (o)$$

The series on the right is a geometrical series of n terms. We have then

$$\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}\,e^{bxi} = \frac{\pi}{e^{nki-bnk}}\cdot\frac{e^{(n+1)ki-\frac{1}{2}(2n+1)bk}-e^{ki-\frac{1}{2}bk}}{e^{k(i-b)}-1}$$

$$= \frac{\pi e^{-\frac{1}{2}bk}}{1+e^{-bk}} = \frac{\pi}{e^{\frac{1}{2}bk}+e^{-\frac{1}{2}bk}},^{**} \qquad (o')$$

$$\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}\cos bx = \frac{\pi}{e^{\frac{1}{2}bk}+e^{-\frac{1}{2}bk}}\ (o_1'); \int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}\sin bx = 0. \quad (o_2')$$

It is also evident that

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* For want of sorts, k is here, and throughout the rest of this article, written instead of the Agate Greek π.—Compositor.

** The same result would be found if in (o) we place $n=1$- It should also be noted that this form satisfies (11) and could be calculated by (12).

$$\int_0^\infty \frac{dx}{e^x+e^{-x}}\cos bx = \frac{\pi}{2(e^{\frac{1}{2}bk}+e^{-\frac{1}{2}bk})}. \qquad (o'_3)$$

Because $\left.\frac{x^p}{e^x+e^{-x}}\right]_{x=\pm\infty} = 0$, any algebraical form may be chosen for $\varphi(z)$ in (m).

2. For the lower sign the roots of (i) are:

$$z_0 = 0;\ z_{\pm 1} = \pm\pi i;\ z_{\pm 2} = 2\pi i \ldots z_{\pm n} = \pm n\pi i \ldots \qquad (p)$$

If we put $y_n = 2\pi$ and $y_0 = 0$, then $z_n = n\pi$ and $z_0 = 0$ coincide with these limits and formulæ $(10''_3)$ and $(10'_3)$ respectively are to be used for these roots, which is one half of what (10) gives. For $z_s = s\pi i$ we have the whole correction by (10),

$$\Delta_{ski} = 2\pi i\varphi(s\pi i)\left[\frac{z-s\pi i}{e^z-e^{-z}}\right]_{z=ski} = 2\pi i\varphi(s\pi i)\frac{1}{e^{ski}+e^{-ski}} = \pi i e^{ski}\varphi(s\pi i). \quad (q)$$

We have then by (h')

$$\int_{-\infty}^{\infty} dx\left[\frac{\varphi(x+n\pi i)}{e^{x+nki}-e^{-x-nki}} - \frac{\varphi(x)}{e^x-e^{-x}}\right] = -\pi i\left\{\tfrac{1}{2}\varphi(0) + \sum_1^{n-1}\left[e^{ski}\varphi(s\pi i)\right] + \tfrac{1}{2}e^{nki}\varphi(n\pi i)\right\},$$

or, since $e^{nki} = e^{-nki}$,

$$\int_{-\infty}^{\infty}\frac{dx}{e^x-e^{-x}}\left[e^{nki}\varphi(x+n\pi i)-\varphi(x)\right] = -\pi i\left\{\tfrac{1}{2}\varphi(0) + \sum_1^{n-1}\left[e^{ski}\varphi(s\pi i)\right] + \tfrac{1}{2}e^{nki}\varphi(n\pi i)\right\}. \quad (r)$$

Let $\varphi(z) = e^{-c^2z^2}$, then as we have already shown the condition for (h') to be true is satisfied and we have

$$\int_{-\infty}^{\infty}\frac{dx}{e^x-e^{-x}}\left[e^{nki-c^2(x+nki)^2} - e^{-c^2x^2}\right] = -\pi i\left[\tfrac{1}{2} + \sum_1^{n-1}(e^{ski+s^2c^2k^2}) + \tfrac{1}{2}e^{nki+n^2c^2k^2}\right]. \ (s)$$

Separating the possible from the impossible, we obtain:

$$\int_{-\infty}^{\infty}\frac{dx e^{-c^2x^2}}{e^x-e^{-x}}\left\{e^{c^2n^2k^2}\cos\left[n\pi(1-2c^2x)\right]-1\right\} = 0, \qquad (s'_1)$$

$$\int_{-\infty}^{\infty}\frac{dx e^{-c^2x^2}}{e^x-e^{-x}}\sin\left[n\pi(1-2c^2x)\right] = -\pi e^{-c^2n^2k^2}\left[\tfrac{1}{2} + \sum_1^{n-1}(e^{ski+s^2c^2k^2}) + \tfrac{1}{2}e^{nki+n^2c^2k^2}\right]. \ (s'_2)$$

Also let $\varphi(z) = e^{bzi}$, then in (r)

$$\int_{-\infty}^{\infty}\frac{dx}{e^x-e^{-x}}\left[e^{nki+b(x+nki)i} - e^{bxi}\right] = -\pi i\left[\tfrac{1}{2} + \sum_n^{n-1}(e^{ski-sbk}) + \tfrac{1}{2}e^{nki-nbk}\right]. \qquad (t)$$

Place $n = 1$, then

$$\int_{-\infty}^{\infty}\frac{e^{bxi}dx}{e^x-e^{-x}} = \frac{\pi i}{e^{-bk}+1}\left(\tfrac{1}{2}-\tfrac{1}{2}e^{-bk}\right)\ (t'), \text{ and } \int_{-\infty}^{\infty}\frac{\cos bx dx}{e^x-e^{-x}} = 0, \qquad (t'_1)$$

$$\int_{-\infty}^{\infty}\frac{\sin bx\, dx}{e^x-e^{-x}} = 2\int_0^{\infty}\frac{\sin bx\, dx}{e^x-e^{-x}} = \frac{\pi}{2}\cdot\frac{e^{bk}-1}{e^{bk}+1}. \qquad (t'_2)$$

This may suffice to show at least a glimpse of the method for the form $z = x+yi$. Other forms may be developed in a similar manner which give rise to new relations. The method requires extreme caution with regard to the conditions to be satisfied by the element function.